A perturbative approach to a nonlinear advection-diffusion equation of particle transport

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Abstract

We explore analytical techniques for modeling the nonlinear cosmic ray transport in various astrophysical environments which is of significant current research interest. While nonlinearity is most often described by coupled equations for the dynamics of the thermal plasma and the cosmic ray transport or for the transport of the plasma waves and the cosmic rays, we study the case of a single but nonlinear advection-diffusion equation. The latter can be approximately solved analytically or semi-analytically, with the advantage that these solutions are easy to use and, thus, can facilitate a quantitative comparison to data. In the present study we extend our previous work on nonlinear diffusion in a threefold manner. First, instead of employing an integral method to the case of pure nonlinear diffusion, we apply an expansion technique to the advection-diffusion equation. We use the technique systematically to analyze the effect of nonlinear diffusion for the cases of constant advection combined with time-varying source functions. Second, we present a alternate formulation for the used nonlinear diffusion coefficient and succesfully apply our expansion technique as well. Third, we extend the study from the one-dimensional, Cartesian geometry to the radially symmetric case, which allows us to treat more accurately the nonlinear diffusion problems on larger scales away from the source.

Basic theory

We begin with the diffusive part of the linear transport equation:

$$\frac{\partial f}{\partial t} = \nabla\left[D\left(r,t\right)\nabla f\right] \tag{1}$$

In linear theories the diffusion coefficient D is only depending on x and t. As shown by e.g. Ptuskin et al.(2008) the diffusion coefficient can also depend on the gradient of the distribution function as well, so that we can take it as:

$$D(f) = D_0\left(\frac{f_0}{r_0}\right)^{\nu}|\nabla f|^{-\nu} \tag{2}$$

The parameter ν represent the nonlinear coefficient, that has been identified for several models, for example as $\nu = \frac{1}{2}$ for moving magnetic mirrors, or $\nu = \frac{2}{3}$ for a Kolmogorov-type energy dissipation of the induced waves in the background plasma. Using this formulation, a monoenergetic 1-D transport equation in a Cartesian geometry would read as:

$$\frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(D_0\left|\frac{\partial f}{\partial x}\right|^{-\nu}\frac{\partial f}{\partial x}\right) + Q_0 \tag{3}$$

It is not possible to derive a solution for this equation, using an arbitrary nonlinear coefficient ν. So our goal is to develop a method to approximate solutions. For small ν one can expand f in orders of ν, so it reads as:

$$f = f_0 + \nu f_1 + \nu^2 f_2 + \ldots \tag{4}$$

Inserting this into the transport equation and ordering the resulting expression in orders of ν we get a set of equations:

$$\begin{aligned}
\mathcal{L}_{\text{cart}} f_0 &= Q_0\\
\mathcal{L}_{\text{cart}} f_1 &= Q_1\\
&\ldots\\
\mathcal{L}_{\text{cart}} f_n &= Q_n\\
\mathcal{L}_{\text{cart}} &= \frac{\partial}{\partial t} + V\frac{\partial}{\partial x} - D_0\frac{\partial^2}{\partial x^2}\\
Q_1 &= -D_0\left(1+\ln|f_{0,x}|\right)f_{0,xx}\\
Q_2 &= -D_0\left(\frac{f_{1,x}}{f_{0,x}} - \frac{1}{2}\ln^2\left(|f_{0,x}|\right) - \ln\left(|f_{0,x}|\right)\right)f_{0,xx}\\
&\quad - D_0\left(1+\ln\left(|f_{0,x}|\right)\right)f_{1,xx}\, etc.
\end{aligned} \tag{5}$$

This formula can be combined with the fundamental solution Γ_{cart} of the operator $\mathcal{L}_{\text{cart}}$, so we only have to calculate the convolutions of said fundamental equation and the sources on the right hand side.

$$\Gamma_{\text{cart}} = \frac{1}{\sqrt{4\pi D_0 t}}\exp\left[-\frac{(x-Vt)^2}{4D_0 t}\right]$$

$$f_n\left(x,t\right) = \int \Gamma_{\text{cart}}\left(x-x_0, t-t_0\right)Q_n\left(x_0,t_0\right)dt_0 dx_0 \tag{6}$$

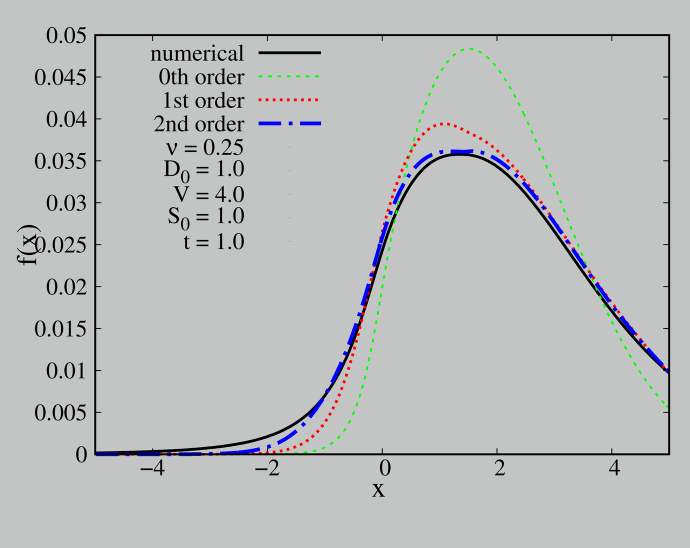
This takes much less computational power, than solving the transport equation directly, while producing decent results, even after only two orders of approximation.

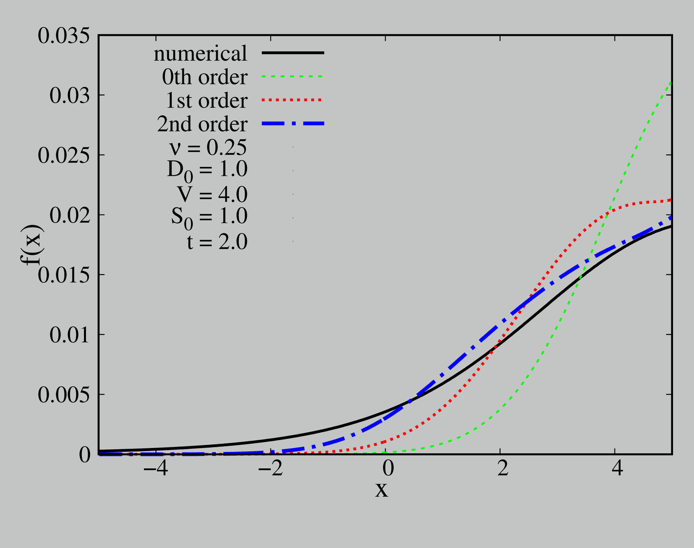
Results for constant advection in a Cartesian geometry

Numerical and semi-analytical results up to the second order of approximation for a source $Q_0 = S_0\delta\left(x\right)\left(t-t^2\right)$ for velocity $V = 4$, sourceparameter $S_0 = 1$ and $\nu = 0.25$:

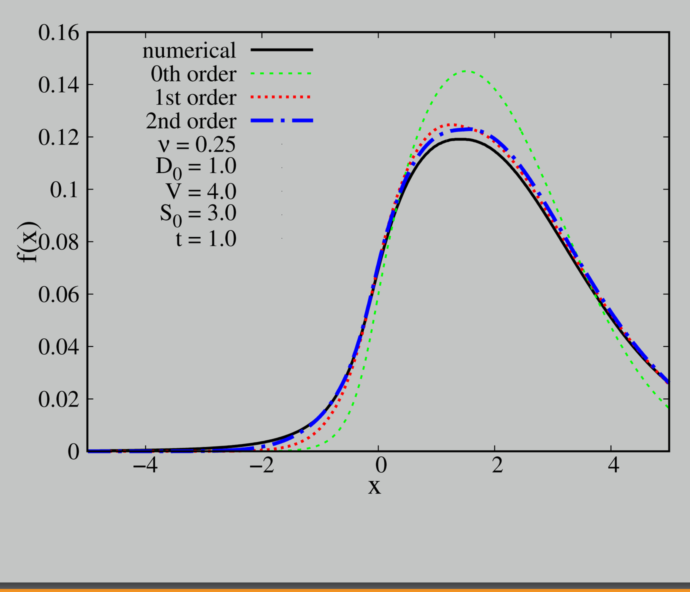
$t = 1$ | $t = 2$

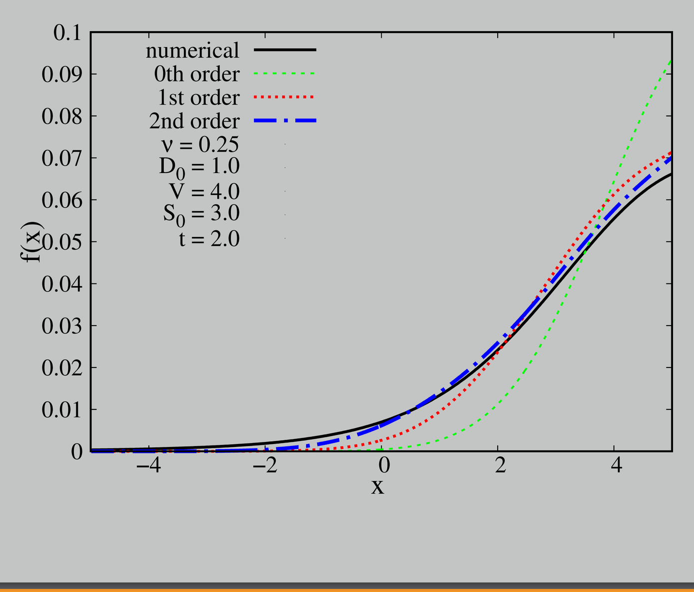
$D_0 = 1$:





$D_0 = 2$:





Variations and different geometries

The nonlinearity introduced in the previous section has the disadvantage of diverging for the case $f_x \to 0$. To avoid this, we can formulate the diffusion in a more consistent fashion $D = \frac{\tilde{D}}{|f_x|^{\nu}+\lambda_0}$. In this case $\tilde{D}$ and λ_0 are constants, who determine the limiting cases for $\nu \to 0$ and $f_x \to 0$. For this scenario we derive the fundamental solution Γ_{cons} and sources:

$$\begin{aligned}
\mathcal{L}_{\text{cons}} &= \frac{\partial}{\partial t} + V\frac{\partial}{\partial x} - \frac{\tilde{D}}{1+\lambda_0}\frac{\partial^2}{\partial x^2}\\
\Gamma_{\text{cons}} &= \frac{1}{\sqrt{4\pi\frac{\tilde{D}}{1+\lambda_0}t}}\exp\left[-\frac{(x-Vt)^2}{4\frac{\tilde{D}}{1+\lambda_0}t}\right]\\
Q_1 &= -\tilde{D}\frac{1+\ln|f_{0,x}|}{(1+\lambda_0)^2}f_{0,xx}, \qquad (7)\\
Q_2 &= -\tilde{D}\frac{\ln|f_{0,x}|+1}{(1+\lambda_0)^2}f_{1,xx} + \tilde{D}\frac{\ln^2|f_{0,x}|+2\ln|f_{0,x}|}{(1+\lambda_0)^3}f_{0,xx}\\
&\quad - \tilde{D}\frac{\frac{1}{2}\ln^2|f_{0,x}|+\ln|f_{0,x}|}{(1+\lambda_0)^2}f_{0,xx} - \tilde{D}\frac{f_{1,x}}{(1+\lambda_0)^2 f_{0,x}}f_{0,xx}. \qquad (8)
\end{aligned}$$

Another possible alteration to the basic formula in the first section is to choose a non Cartesian geometry. For many astrophysical applications the spherical geometry is the most usable geometry. The transport equation then reads as:

$$\frac{\partial f}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D\left(f\right)\frac{\partial f}{\partial r}\right) + Q_0 \tag{9}$$

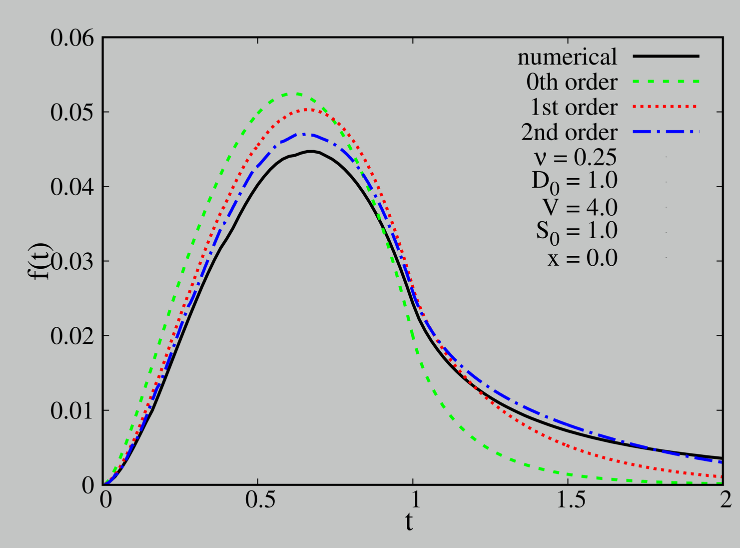
Treating this equation in the same way as before, we get a set of equations and a linear operator $\mathcal{L}_{\text{rad}}$, that can be solved semi analytically.

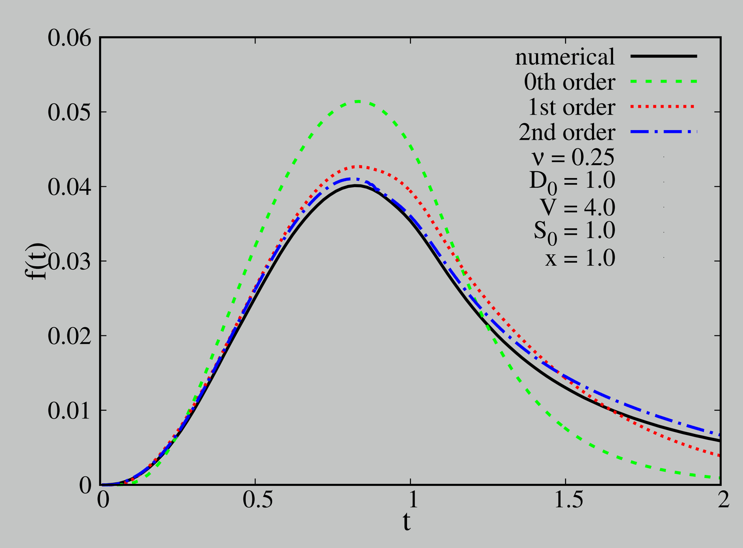
$$\begin{aligned}
\mathcal{L}_{\text{rad}} &= \frac{\partial}{\partial t} - D_0\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\\
\mathcal{L}_{\text{rad}} f_n &= Q_n\\
Q_1 &= -\left(1+\ln\left(|f_{0,r}|\right)\right)f_{0,rr} - \frac{1}{r}\ln\left(|f_{0,r}|\right)f_{0,r}\\
Q_2 &= -\frac{2}{r}\ln\left(|f_{0,r}|\right)f_{1,r} - \frac{2}{r}f_{1,r} - \ln\left(|f_{0,r}|\right)f_{1,rr} - \frac{1}{f_{0,r}}f_{1,r}f_{0,rr}\\
&\quad - f_{1,rr} + \ln\left(|f_{0,r}|\right)f_{0,rr} + \frac{\ln^2\left(|f_{0,r}|\right)}{2}f_{0,rr} + \ln^2|f_{0,r}|\frac{f_{0,r}}{r}
\end{aligned} \tag{10}$$

For solving this equations we use a modified version of the Greens formula of Webb and Gleeson(1977), that reads:

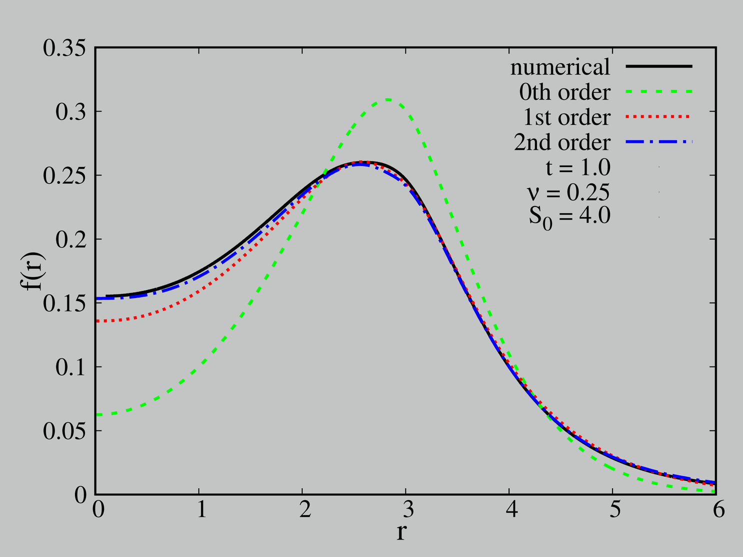
$$\begin{aligned}
f\left(r,t\right) &= \frac{1}{r^2}\int_0^t dt'\int_0^\infty dr' r'^2 Q\left(r',t'\right)G\left(r',r,t-t'\right)\\
G\left(r',r,t-t'\right) &= \frac{\frac{r}{r'}}{2\sqrt{\pi\left(t-t'\right)}}\\
&\quad\times\left(\exp\left(-\frac{\left(r'-r\right)^2}{4\left(t-t'\right)}\right) - \exp\left(-\frac{\left(r'+r\right)^2}{4\left(t-t'\right)}\right)\right)
\end{aligned} \tag{11}$$

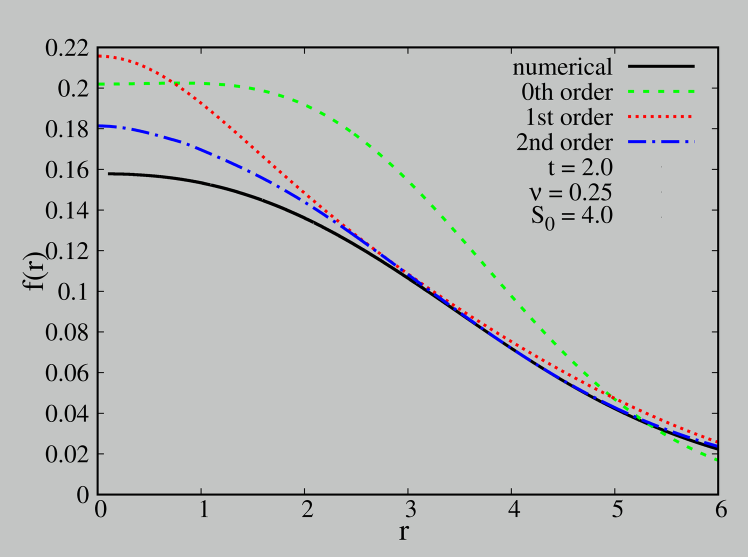
Results for modified scenarios





Solution for a consistent diffusion coefficient up to the second order, for a pulse like source. The parameters are $D_0 = 1$ and $\tilde{D} = 10^3$ with $V = 4$ for $x = 0$ and $\nu = \frac{1}{4}$(left), $x = 1$ and $\nu = \frac{1}{4}$(right).





Solution for a spherical geometry up to the second order, for a pulse like source. The nonlinearity parameter was chosen to be $\nu = \frac{1}{4}$ with the source strength S chosen as $S_0 = 4$. The plotting times are $t = 1$ (left) and $t = 2$ (right).

Discussion and Outlook

We derived a seminanalytical formula to derive solutions to a distinct form of nonlinear diffusion advection equations. We presented a selected number of results of scenarios of different geometries and implementations of the diffusion coefficient, comparing them to much more time intensive numerical results. We can see, that for small nonlinear parameters ν the formula gives decent results in just two approximation steps. The quality of the approximations is dependent on the paramter ν. The results obtained for the monoenergetic transport equation can be used as the groundwork for shock acceleration and non-constant velocity profiles. For more examples and scenarios you can take a look at Walter et al.(2020, Phys.Pl.27,id.082901).